Lecture 14. Spans and subspaces

Def Let V be a vector space.

(1) A subspace of V is a vector space contained in V (with the same addition and scalar multiplication)

(2) A linear combination of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ in V is a vector of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ with $c_1, c_2, \cdots, c_n \in \mathbb{R}$.

(3) The span of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ in V is the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

(4) Vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ in V are linearly independent if we have $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \neq \vec{0}$ unless $c_1, c_2, \cdots, c_n$ are all zero.

Prop Given vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ in a vector space V, their span is a subspace of V.

pf

- zero vector: $\vec{0} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \cdots + 0\cdot\vec{v}_n$ in the span
- closed under addition:

  $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ in the span

  $\Longrightarrow \vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_n + d_n)\vec{v}_n$ in the span
- closed under scalar multiplication:

  $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ in the span

  $\Longrightarrow c\vec{u} = cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_n\vec{v}_n$ in the span for any $c \in \mathbb{R}$

Hence the span of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is a subspace of V.

Ex Determine whether each set is a subspace of $\mathbb{R}^2$.

(1) The set of all vectors of the form $\begin{bmatrix} a-b \\ 2 \end{bmatrix}$ with $a, b \in \mathbb{R}$

Sol The set does not contain the zero vector.

(the second coordinate can never be 0)

$\Rightarrow$ The set is $\boxed{\text{not a subspace of } \mathbb{R}^2}$

(2) The set of all vectors of the form $\begin{bmatrix} 2a-3b \\ a+4b \end{bmatrix}$ with $a, b \in \mathbb{R}$

Sol We may write $\begin{bmatrix} 2a-3b \\ a+4b \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$\Rightarrow$ The set is the span of $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

$\Rightarrow$ The set is $\boxed{\text{a subspace of } \mathbb{R}^2}$

(3) The set of all vectors of the form $\begin{bmatrix} a^3-b^3 \\ 2b^3 \end{bmatrix}$ with $a, b \in \mathbb{R}$.

Sol We may write $\begin{bmatrix} a^3-b^3 \\ 2b^3 \end{bmatrix} = a^3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b^3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\Rightarrow$ The set is the span of $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(the scalar factors a^3 and b^3 are arbitrary)

$\Rightarrow$ The set is $\boxed{\text{a subspace of } \mathbb{R}^2}$

(4) The set of all vectors of the form $\begin{bmatrix} a^2-b^2 \\ 2b^2 \end{bmatrix}$ with $a, b \in \mathbb{R}$.

Sol The set is not closed under scalar multiplication:

$\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in the set with $a=1$ and $b=1$

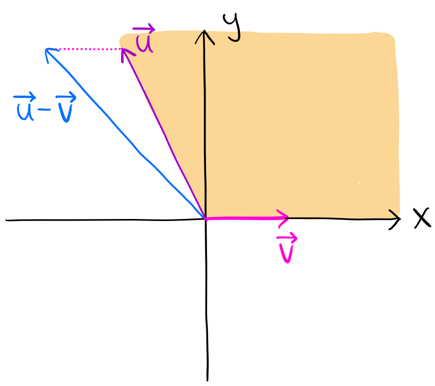
$-\vec{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ not in the set

(the second coordinate $2b^2$ can never be negative)

$\Longrightarrow$ The set is $\boxed{\text{not a subspace of } \mathbb{R}^2}$

Note Although we may write $\begin{bmatrix} a^2-b^2 \\ 2b^2 \end{bmatrix} = a^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b^2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the set is not spanned by $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. In fact, the scalar factors a^2 and b^2 are not arbitrary as they can never take negative values.

* $\vec{u} - \vec{v}$ does not lie in the set even though it lies in the span of $\vec{u}$ and $\vec{v}$.



the set is given by nonnegative combinations of $\vec{u}$ and $\vec{v}$

Ex Determine whether each set is a subspace of $\mathbb{P}$, where $\mathbb{P}$ denotes the space of all polynomials.

(1) The set of all polynomials of the form $p(t) = at^2$ with $a \in \mathbb{R}$.

Sol The set is the span of the polynomial t^2.

$\Rightarrow$ The set is $\boxed{\text{a subspace of } \mathbb{P}}$

(2) The set of all polynomials of the form $p(t) = a + bt$ with $a, b \in \mathbb{R}$.

Sol We may write $a + bt = a \cdot 1 + b \cdot t$

$\Rightarrow$ The set is the span of the polynomials 1 and t.

$\Rightarrow$ The set is $\boxed{\text{a subspace of } \mathbb{P}}$

Note Here we regard 1 as a constant polynomial $(1 = t^0)$

(3) The set of all polynomials of the form $p(t) = a + t$ with $a \in \mathbb{R}$.

Sol The set does not contain the zero polynomial.

$(0 \neq a + t$ for any $a \in \mathbb{R})$

$\Rightarrow$ The set is $\boxed{\text{not a subspace of } \mathbb{P}}$

Note The set is not spanned by 1 and t. In fact, we have

$a + t = a \cdot 1 + 1 \cdot t$ where the scalar factor for t is $\underbrace{\text{fixed to be } 1}_{\text{not arbitrary}}$.

* $3 + 2t$ does not lie in the set even though it lies in the span of 1 and t.

(4) The set of all polynomials $p(t)$ with $p(3) = 0$.

Sol
- zero vector : the $\underbrace{\text{zero polynomial}}_{\text{zero function}}$ in the set

- closed under addition :

  $p(t)$ and $q(t)$ with value 0 at $t = 3$

  $\Rightarrow p(t) + q(t)$ with value $0 + 0 = 0$ at $t = 3$

- closed under scalar multiplication :

  $p(t)$ with value 0 at $t = 3$

  $\Rightarrow cp(t)$ with value $c \cdot 0 = 0$ at $t = 3$ for any $c \in \mathbb{R}$

Hence the set is $\boxed{\text{a subspace of } \mathbb{P}}$

(5) The set of all polynomials $p(t)$ with $p(3) = 4$

Sol The set does not contain the zero polynomial.

$\Rightarrow$ The set is $\boxed{\text{not a subspace of } \mathbb{P}}$